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TOWARDS A MORE GENERAL NOTION OF GELFAND–KIRILLOV DIMENSION

BY

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ABSTRACT

Gelfand–Kirillov dimension (GK) has proved to be a useful invariant for algebras over fields. In this paper we generalize the notion of GK to algebras over commutative Noetherian rings by replacing vector space dimension with reduced rank. It turns out that most results about GK have analogues for the new GK.

1. Introduction

Gelfand–Kirillov dimension (GK) has been an important invariant in the theory of algebras over a field for the past twenty years. Although it is rarely exact, it has the advantage over Krull dimension of being both symmetric and ideal invariant. It has been applied successfully to enveloping algebras, Weyl algebras, and more generally to filtered and graded algebras. The principal objective here is to extend the notion of Gelfand–Kirillov dimension to algebras over commutative

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Noetherian rings; in fact, our techniques may be applied to algebras over an even wider class of base rings, which includes integral domains. Ideally, of course, one would like a notion of GK that worked for algebras over an arbitrary commutative ring, but that does not seem to be possible.

The generalization we offer here is based on replacing vector space dimension with reduced rank. The very direct analogy with the "classical" GK allows us to generalize many results and proofs. The main drawback is that this notion of dimension is not sensitive to the presence of torsion within the algebra, a difficulty inherent in the choice of reduced rank as a starting point. It should also be mentioned that what is measured is the size of the algebra as a module over the chosen base ring rather than its absolute size (which presumably should take into account the "size" of the homomorphic image of the base ring present in the algebra). Strictly speaking, this is not in conflict with the classical GK dimension, which gives no information as to the field involved, but it does seem to cause difficulties when interpreting the meaning of dimension 1 for instance.

This paper is organized as follows: Section 2 fixes our notation and introduces $\widehat{\mathrm{GK}}$, our generalized notion of GK, and its basic properties. Further, we focus on the special case where the algebra in question satisfies a polynomial identity. Section 3 describes how $\widehat{\mathrm{GK}}$ behaves when the base ring is changed. We cover localization and finite extensions of the base ring, and also relate $\widehat{\mathrm{GK}}$ to the ordinary GK when the base ring is an affine algebra over a field. In Section 4 we turn to $\widehat{\mathrm{GK}}$ for modules. It turns out to be just as simple to devise this as it is in the classical situation. Section 5 contains a number of applications. We give results on group algebras, prime ideals in ring extensions, Ore extensions, and enveloping algebras.

2. Algebras over commutative Noetherian rings

Given an affine algebra over a field, $A = k\{a_1, \ldots, a_n\}$ say, $\mathrm{GK}_k(A)$ is computed as

$$\mathrm{GK}_k(A) = \limsup_n \log_n(\dim_k V^n),$$

where $V = 1 \cdot k + a_1 k + \cdots + a_n k$ and $\log_n(x) = \frac{\log x}{\log n}$.

It is well known that $\mathrm{GK}_k(A)$ does not depend on the choice of generators of the algebra, see e.g [4, Lemma 1.1].

The crucial thing in the definition is that vector space dimension is available.

Assuming that the base ring is just a commutative (Noetherian) ring, one is faced with the challenge of finding a suitable replacement for dimension or, somehow, reducing to a case where "classical" GK dimensions can be calculated. As stated in the introduction, the approach taken here will be to follow the first course and replace dimension with reduced rank, but later it will be clear that our generalization can also be computed as a supremum of certain classical GKs.

Let us first fix some notation. Throughout, C denotes a commutative ring and A a C–algebra which is not necessarily affine. We use the following notation when dealing with algebras: $C[a_i]$ denotes a commutative algebra, $C\{a_i\}$ a not necessarily commutative algebra, and $C\langle a_i\rangle$ a free algebra.

We use the notation $\rho = \rho_R$ to denote the reduced rank over a ring R which is either clear from context or is indicated as a subscript. We assume the basic properties of reduced rank — see e.g. [7] for a description that is more than ample for our purposes. Also, let $\log_n(x) = \frac{\log x}{\log n}$, and extend this definition to yield $\log_n(0) = -\infty$.

Definition 2.1: Consider an affine C-algebra $A = C\{a_1, \ldots, a_n\}$, where C has the property that reduced rank is defined and finite for finitely generated modules. Define

$$\widehat{\mathrm{GK}}_C(A) = \limsup_n \log_n(\rho_C(M^n)), \tag{1}$$

where $M = 1 \cdot C + a_1 C + \cdots + a_n C$. For non-affine algebras we define

$$\widehat{\mathrm{GK}}_C(A) = \sup_{A'} \widehat{\mathrm{GK}}_C(A'), \tag{2}$$

where the supremum is taken over all affine subalgebras A' of A.

This definition makes sense for all commutative rings having nilpotent prime radical that are Goldie modulo their prime radical. However, we shall be a little more restrictive in what follows.

Throughout the sequel C will denote a commutative base ring for which the prime radical, $N(C)$, is a finitely generated ideal and $C/N(C)$ is Goldie, e.g. C Noetherian, an integral domain, or has an Artinian ring of quotients.

In the case where C is reduced, an obvious reduction may be made; for in this case we have

$$\rho_C(M^n) = \mathrm{length}_Q(M^n \otimes Q)$$

where Q is the semisimple quotient ring of C, and so

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_Q(A \otimes Q).$$

In fact, this sort of reduction may be carried out over all base rings under consideration as follows. Let $N = N(C)$, $N = n_1 C + \cdots + n_s C$, and suppose that $N^k = 0$. Now let M denote a (right) C-module. There is a natural epimorphism

$$M^s \twoheadrightarrow MN$$

given by $(m_1, \ldots, m_s) \mapsto \sum m_i n_i$, and hence we have epimorphisms

$$M^{s^r} \twoheadrightarrow MN^r.$$

These homomorphisms induce epimorphisms

$$(M/MN)^{s^{r-1}} \twoheadrightarrow MN^{r-1}/MN^r,$$

and hence we have

$$\begin{aligned}\rho_{C/N}(M/MN) \leq \rho_C(M) &= \rho_C(M/MN) + \cdots + \rho_C(MN^{k-1}/MN^k) \\ &\leq (1 + s + \cdots + s^{k-1})\rho_C(M/MN) \text{ constant } \cdot \rho_{C/N}(M/MN).\end{aligned}$$

These inequalities and the definition imply that

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_{C/N}(A/AN)$$

for all (affine) C-algebras — or phrased differently: The "obvious" nilpotent ideal in A can be disregarded.

Letting $Q = Q(C/N)$ be the classical ring of quotients of C/N we see that the upshot of all this is that

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_Q(A/AN \otimes_{C/N} Q),$$

and this eases the computational pain quite a bit.

Let us study the right hand side of this last equality a bit further. Note that the quotient ring Q is a direct sum of fields, $Q = L_1 \oplus \cdots \oplus L_t$ say, and denote the corresponding minimal idempotents in Q by $e_1, \ldots, e_t$. Appealing to the definition of $\widehat{\mathrm{GK}}$ we find a bound

$$\max\{\mathrm{GK}_{L_i}(Be_i)\} \leq \widehat{\mathrm{GK}}_Q(B)$$

for any Q-algebra, B.

In fact this is an equality. To see the reverse inequality we employ an argument similar to [4, Proposition 3.2]. Retaining the above notation, let V be a finitely generated generating submodule for B. Then Ve_i is a generating subspace for Be_i. Further let $d(n) = \text{length}_Q V^n$ and $d_i(n) = \dim_{L_i} V^n e_i$. We may assume that $\alpha = \max\{\text{GK}_{L_i}(Be_i)\} < \infty$ since otherwise the desired inequality is trivial. Now, for any $\varepsilon > 0$ we have $d_i(n) \leq n^{\alpha+\varepsilon/2}$ for large n, and hence

$$d(n) = \sum_i d_i(n) \leq t \cdot n^{\alpha+\varepsilon/2} \leq n^{\alpha+\varepsilon}$$

for large enough n, so that $\widehat{\text{GK}}_Q(B) \leq \alpha$. This completes the proof of

PROPOSITION 2.2: *Let A be an affine C-algebra. Then*

$$\widehat{\text{GK}}_C(A) = \max_i\{\text{GK}_{Q(C/P_i)}(A/AP_i \otimes Q(C/P_i))\},$$

where the maximum is taken over all minimal prime ideals in C.

This direct analogue of the classical definition has its serious drawbacks, e.g. $\widehat{\text{GK}}_{\mathbb{Z}}(\frac{\mathbb{Z}}{2}\langle x,y\rangle) = -\infty$, or, more generally, any $\mathcal{C}(N(C))$-torsion algebra has $\widehat{\text{GK}}$ $-\infty$. This is of course a defect which makes $\widehat{\text{GK}}$ ill suited for some of the intended applications, e.g. a "GK 1 implies PI" theorem [10] is in general out of the question. There are, however, also a number of positive results about $\widehat{\text{GK}}$, which we now review. The proofs here are easily obtained by mimicking the proofs of the corresponding claims for GK, for which we refer the reader to [4].

PROPOSITION 2.3:

- *Let A be an affine C-algebra. Then $\widehat{\text{GK}}$ does not depend on the choice of generators.*
- *If $A \subseteq B$ are C-algebras then $\widehat{\text{GK}}_C(A) \leq \widehat{\text{GK}}_C(B)$, and if I is an ideal in a C-algebra A then $\widehat{\text{GK}}_C(A/I) \leq \widehat{\text{GK}}_C(A)$.*
- *If $A \subseteq B$ are C-algebras and B_A is a finitely generated module then $\widehat{\text{GK}}_C(A) = \widehat{\text{GK}}_C(B)$.*
- *Let A be an affine C-algebra. Then $\widehat{\text{GK}}_C(A) \leq 0$ if and only if $\rho_C(A) < \infty$.*
- *Let A be a C-algebra. Then $\widehat{\text{GK}}_C(A) > 0$ implies $\widehat{\text{GK}}_C(A) \geq 1$.*

PROPOSITION 2.4: *Let A be an affine C-algebra and let I be an ideal in A which contains a (right) regular element $c \in A$. Then*

$$\widehat{\text{GK}}_C(A) \geq 1 + \widehat{\text{GK}}_C(A/I).$$

Proof: If $I \cap \mathcal{C}_C(N) \neq \varnothing$, then we immediately have $\widehat{\mathrm{GK}}_C(A/I) = -\infty$, and so the assertion follows. Consequently we may assume that $I \cap \mathcal{C}_C(N) = \varnothing$.

Now set $\overline{A} = A/AN$ and $\overline{I} = I + AN/AN$. Further, let $B = \overline{A} \otimes Q$, where $Q = Q(C/N)$ is the semisimple quotient ring of C/N, and let $\overline{A}/\overline{I} \otimes Q \simeq \overline{A} \otimes Q/\overline{I} \otimes Q = B/J$, where J is a proper two sided ideal of B. Note that B is an affine Q-algebra, and that J contains a (right) regular element of B.

Since $\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_Q(B)$ and $\widehat{\mathrm{GK}}_C(A/I) = \widehat{\mathrm{GK}}_Q(B/J)$ it suffices to prove that

$$\widehat{\mathrm{GK}}_Q(B) \geq \widehat{\mathrm{GK}}_Q(B/J) + 1,$$

and this is carried out along the lines of [4, Proposition 3.15].

We fix some notation: Let $B = Q\{b_1, \ldots, b_k\}$ and let $c \in J$ be a (right) regular element. Set $M = 1 \cdot Q + c \cdot Q + b_1 \cdot Q + \cdots + b_k \cdot Q$ and $\overline{M} = M + J/J$.

Since Q is semisimple we have

$$(M^n \cap J) \oplus D_n = M^n$$

where

$$D_n \simeq \frac{M^n}{M^n \cap J} \simeq \frac{M^n + J}{J} \simeq \overline{M}^n$$

and

$$D_n \cap cB = 0.$$

We immediately conclude that the sum $D_n + cD_n + \cdots + c^n D_n$ (in B_Q) is direct and hence that

$$D_n \oplus cD_n \oplus \cdots \oplus c^n D_n \subseteq M^{2n}.$$

Now let $d(n) = \mathrm{length}_Q\, M^n$ and $\overline{d}(n) = \mathrm{length}_Q\, \overline{M}^n$. Then

$$d(2n) = \mathrm{length}_Q\, M^{2n} \geq n \cdot \mathrm{length}\, D_n = n \cdot \mathrm{length}_Q\, \overline{M}^n = n \cdot \overline{d}(n)$$

and hence

$$\begin{aligned} \widehat{\mathrm{GK}}_Q(B/J) + 1 &= 1 + \limsup \log_n \overline{d}(n) \\ &\leq \limsup(\log_n \overline{d}(n) + \log_n n) \\ &= \limsup(\log_n n \cdot \overline{d}(n)) \\ &\leq \limsup \log_n d(2n) \\ &= \limsup \log_n d(n) = \widehat{\mathrm{GK}}_Q(B) \end{aligned}$$

as desired. ∎

While we will apply this result to Ore Extensions in Section 5.4, it does not have the far reaching consequences that the corresponding result for GK does. For example, one would like to apply the result to the regular sequences of a commutative algebra, but this simply does not work, the problem being that it takes very little to drop $\widehat{\mathrm{GK}}$ to 0, or even to $-\infty$. For instance, consider $\mathbb{Z}[x]$ which has $\widehat{\mathrm{GK}}_{\mathbb{Z}}\mathbb{Z}[x] = 1$ but a regular sequence 2, x.

We do at least get the following (expected) corollaries, but the possibility of $\widehat{\mathrm{GK}}$ being $-\infty$ will limit their effectiveness.

COROLLARY 2.5: *Let A be an affine C-algebra with all prime factors Goldie (e.g. A Noetherian). If P is a prime ideal of A, then*

$$\widehat{\mathrm{GK}}_C(A) \geq \widehat{\mathrm{GK}}_C(A/P) + \mathrm{ht}(P).$$

If in addition C is Artinian, then $\widehat{\mathrm{GK}}_C(A) \geq \mathrm{ht}(P)$ for all prime ideals P of A.

Proof: The first statement follows from the Proposition just as in [4, Corollary 3.16]. The second claim is valid since over an Artinian base ring, $\widehat{\mathrm{GK}}$ is always greater than or equal to zero. ∎

COROLLARY 2.6: *Let A be a Noetherian, affine C-algebra, Q a prime ideal of A, and I an ideal of A which properly contains Q. If $Q \cap C = (0)$, then $\widehat{\mathrm{GK}}_C(A/I) < \widehat{\mathrm{GK}}_C(A/Q)$.*

Proof: We may assume that A is a prime, whence $\mathcal{C}_C(N(C)) = C \setminus \{0\}$. Since I is a nonzero ideal of A, it contains a regular element. The result now follows immediately from Proposition 2.4. ∎

Now that we have the basic axiomatic properties of $\widehat{\mathrm{GK}}$ down, it is worthwhile noting that they do not make $\widehat{\mathrm{GK}}$ a dimension function in the sense of Borho [1]. The problem is his condition that $d(M) \geq 0$ for $M \neq 0$ does not hold, and even though we formally has his "dropping condition" it's not as powerful without positivity of the dimension.

We note two more immediate consequences of the reduction of $\widehat{\mathrm{GK}}$ to a maximum of finitely many ordinary GK's. First of all, we can say something very concrete about $\widehat{\mathrm{GK}}$ for PI algebras.

COROLLARY 2.7: *If A is an affine C-algebra satisfying a polynomial identity, then $\widehat{\mathrm{GK}}_C(A) < \infty$. If furthermore A is Noetherian, then $\widehat{\mathrm{GK}}_C(A)$ is an integer.*

Proof: The first claim follows readily from 2.2 and the corresponding result for ordinary GK which was proved by Berele (see [4, Corollary 10.7]). As for the second claim see [4, Corollary 10.16]. Do note, though, that $\widehat{\mathrm{GK}}_C(A)$ is not necessarily equal to the Krull dimension of A, but it is of course the maximum of the Krull dimensions of $A/AP_i \otimes Q(C/P_i)$. ∎

Secondly, there is Lorenz–Small type result about reducing to prime factors in the Noetherian PI case:

COROLLARY 2.8: *Suppose A is an affine, Noetherian C-algebra with prime radical $N(A)$. If A is PI, then*

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_C(A/N(A)) = \max\{\widehat{\mathrm{GK}}_C(A/P)\},$$

where P runs through the minimal primes of A.

Proof: Let $P_1, \ldots, P_n$ denote the minimal prime ideals of C and let $L_i = Q(C/P_i)$. Then, by 2.2, we have

$$\widehat{\mathrm{GK}}_C(A) = \max\{\mathrm{GK}_{L_i}(A/N(C)A \otimes L_i)\}.$$

Now clearly $N(A)/N(C)A \subseteq N(A/N(C)A \otimes L_i)$, so by [6] we have

$$\mathrm{GK}_{L_i}(A/N(C)A \otimes L_i) = \mathrm{GK}_{L_i}(A/N(A) \otimes L_i),$$

and using once more one of the basic reductions we see that

$$\begin{aligned} \mathrm{GK}_{L_i}(A/N(A) \otimes L_i) &\leq \widehat{\mathrm{GK}}_{C/N(C)}(A/N(A) \otimes Q(C/N(C))) \\ &= \widehat{\mathrm{GK}}_C(A/N(A)), \end{aligned} \tag{3}$$

and the first equality follows. The second follows easily, as in [6], from the embedding $A/N(A) \hookrightarrow \oplus A/P$ where P runs through the minimal primes of A. ∎

To conclude this section, we mention two classical results which, while not true in general, are valid if we make some further assumptions on our algebra.

PROPOSITION 2.9: *Let C be a commutative Artinian ring and A an affine, Noetherian PI algebra. Then $\widehat{\mathrm{GK}}_C(A) = |A|$, the classical Krull dimension of A.*

Proof: The main point here is that for an Artinian base ring C with prime radical $N = N(C)$ we have

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_{C/N}(A/AN),$$

and C/N is a direct sum of fields, $C/N = L_1 \oplus \cdots \oplus L_k$, with corresponding idempotents e_i say. The calculation of $\widehat{\mathrm{GK}}$ then reduces to simply

$$\widehat{\mathrm{GK}}_C(A) = \max_i \{\mathrm{GK}_{L_i}((A/AN) \cdot e_i)\}$$

by Proposition 2.2.

The claim now follows from the analogous result for GK [4, Corollary 10.16] by noting that $|A| = \max_i |Ae_i| = \max_i |(A/AN) \cdot e_i|$. ∎

Note that the hypotheses in the next result are automatically satisfied if either the algebra A is prime (and faithful), or the base ring C is Artinian.

PROPOSITION 2.10: *Suppose that A is an affine C-algebra such that A/PA is a torsionfree C/P-algebra for each minimal prime P of C. If $\widehat{\mathrm{GK}}_C(A) \leq 1$, then A is PI.*

Proof: Employing the notation of Corollary 2.8, the hypotheses guarantee that $A/AP \hookrightarrow A/AP \otimes Q(C/P)$ for all minimal prime ideals P in C, and that in each case $\mathrm{GK}_{Q(C/P)}(A/AP \otimes Q(C/P)) \leq 1$. It follows then from [10, Theorem] that each A/AP is PI.

Now we also have the embedding $A/\cap(AP) \hookrightarrow \bigoplus A/AP$, whence $A/\bigcap(AP)$ is PI. As $\bigcap(AP)$ is nilpotent, we conclude that A is PI. ∎

3. Localization and change of base ring

The results obtained so far indicate that results involving assumptions on $\widehat{\mathrm{GK}}$ can often be established with the additional hypothesis that C is an integral domain and then, in many cases, lifted back to the reduced case, and so all the way to the general case. In an effort to extend these reduction techniques, we now consider central localization in the algebra, as well as the effect of changing the base ring to a related ring.

The first result is closely related to [4, Proposition 4.2].

PROPOSITION 3.1: *Let C be reduced and A a C-algebra. Suppose that S is multiplicatively closed subset of regular elements of $Z(A)$ — the center of A. Then*

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_C(A_S).$$

Proof: The general case follows easily from the special case where C is an integral domain. If C is an integral domain we have $\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_k(A \otimes k)$ where k is the quotient field of C by Proposition 3.2 whence we are done by [4, Proposition 4.2]. ∎

Unlike the classical case, it is possible to localize with respect to a multiplicatively closed subset S of the base ring C. Proposition 2.2 immediately implies

PROPOSITION 3.2: *Let C be reduced and A an affine C-algebra. Suppose that S is multiplicatively closed subset of regular elements of C. Then*

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_{C_S}(A_S).$$

Pushing this idea in a slightly different direction, we compare the dimensions of a single algebra computed over related central subrings.

LEMMA 3.3: *Let C be an integral domain which is module finite over a subring C_0. Further, let A be an affine C-algebra. Then*

$$\widehat{\mathrm{GK}}_{C_0}A = \widehat{\mathrm{GK}}_C A.$$

Proof: Let L and L_0 denote the quotient fields of C and C_0. Clearly $m = \dim_{L_0} L < \infty$ and $L = C \otimes_{C_0} L_0$.

Notice that A is an affine algebra over C_0 and let V be a finite dimensional C_0-generating subspace for A containing 1. Of course V is a C-generating subspace as well. If $\{v_i \otimes 1\}$ is a basis for $V^n \otimes_C L$ then $\{v_i \otimes 1\}$ is a linearly independent set in $V^n \otimes_{C_0} L_0$, thus

$$\dim_{L_0} V^n \otimes_{C_0} L_0 \geq \dim_L V^n \otimes_C L.$$

Hence $\widehat{\mathrm{GK}}_{C_0}A \geq \widehat{\mathrm{GK}}_C A$.

Conversely, we have that

$$\dim_{L_0} V^n \otimes_{C_0} L_0 \leq m \cdot \dim_L V^n \otimes_C L,$$

and this implies $\widehat{\mathrm{GK}}_{C_0}A \leq \widehat{\mathrm{GK}}_C A$. ∎

Remark: Actually Lemma 3.3 is true for reduced rings C and C_0 with an analogous argument.

PROPOSITION 3.4: *Let A be a prime, affine, faithful C-algebra, where C is an affine algebra over a field k. Then*

$$\mathrm{GK}_k A \geq \widehat{\mathrm{GK}}_C A + \mathrm{GK}_k C,$$

and if A is PI we have "=".

Proof: Notice that C is an integral domain and that $\mathrm{GK}_k C = \mathrm{trdeg}_k C = |C| = n$.

By the Noether Normalization Lemma we have $R = k[t_1, \ldots, t_n] \subseteq C$ and C is a finitely generated R-module. Now Lemma 3.3 gives that $\widehat{\mathrm{GK}}_C A = \widehat{\mathrm{GK}}_R A$. Thus our claim is

$$\mathrm{GK}_k A \geq \widehat{\mathrm{GK}}_R A + n.$$

Let $A = R\{v_1, \ldots, v_l\}$ and $V = 1 \cdot R + v_1 R + \cdots + v_l R$. Further let

$$\widetilde{V} = 1 \cdot k + v_1 k + \cdots + v_l k + t_1 k + \cdots + t_n k$$

and

$$\varphi(m) = \dim_{k(t_1, \ldots, t_n)} V^m \otimes_R k(t_1, \ldots, t_n).$$

A naive estimate yields [4, p. 10]

$$\dim_k \widetilde{V}^{2m} \geq \varphi(m) \binom{m-1+n}{n-1}$$

from which it follows that $\mathrm{GK}_k A \geq \widehat{\mathrm{GK}}_R A + n$.

For the second claim we recall (see e.g. [9]) that

$$\mathrm{GK}_k A = \mathrm{trdeg}_k Z(Q(A))$$

where $Q(A)$ is the Goldie quotient ring of A. Now the situation is such that

$$k \subseteq k(t_1, \ldots, t_n) \subseteq Z(Q(A))$$

and the claim amounts to the fact that transcendence degrees add up. ∎

It is well known that equality is not true in general, even if C is a field. Indeed, let k be a field, $K = k(t)$, and

$$A = K\langle x, y\rangle / (tx - x^2, ty - y^2).$$

Then $\mathrm{GK}_k K = 1$, $\mathrm{GK}_K A = 1$, and $\mathrm{GK}_k A = \infty$.

4. $\widehat{\mathrm{GK}}$ for modules and filtered/graded techniques

We define $\widehat{\mathrm{GK}}$ for modules mimicking the usual definition of GK for modules.

Definition 4.1: Let $A = C\{a_1, \dots, a_n\}$ and let $M = m_1A + \cdots + m_lA$ be a finitely generated A-module. Further let $V = 1 \cdot C + a_1C + \cdots + a_nC$, and $M_0 = m_1C + \cdots + m_lC$. Now define

$$\widehat{\mathrm{GK}}_C(M) = \limsup_n \log_n(\rho_C(M_0V^n)).$$

For modules not necessarily finitely generated let $\widehat{\mathrm{GK}}_C(M) = \sup \widehat{\mathrm{GK}}_C M'$, where the supremum is over all finitely generated submodules of M.

Dealing with modules instead of algebras raises the question of whether the two dimensions agree "when they should." To be precise: Consider an inclusion of C-algebras $A \subseteq B$. It does in fact follow fairly directly from the definitions that $\widehat{\mathrm{GK}}_C B = \widehat{\mathrm{GK}}_C\ (B_A)$.

In analogy with Proposition 2.2 one can reduce the calculation of $\widehat{\mathrm{GK}}_C M$ to a maximum of ordinary GK's. Following the proof given there we get

PROPOSITION 4.2: *With the notation of Definition 4.1 we have*

$$\widehat{\mathrm{GK}}_C M = \max_i \mathrm{GK}_{Q(C/P_i)}\ (M/MP_i \otimes_{C/N} Q(C/P_i))$$

where P_i denotes the minimal primes of C.

We now note a couple of results that are analogous to well known theorems for GK. They will be useful in applications. The original proofs work because localization is an exact functor and reduced rank is additive over short exact sequences. We refer to [4, Chapter 5] for details.

PROPOSITION 4.3: *Let A be an affine algebra over a commutative Noetherian ring C, and let M and M_i be finitely generated right A-modules. Then*

- $\widehat{\mathrm{GK}}_C(\oplus_{i=1}^r M_i) = \max_i \widehat{\mathrm{GK}}_C(M_i)$.
- *For $0 \to M_1 \to M \to M_2 \to 0$ we have*

$$\widehat{\mathrm{GK}}_C(M) \geq \max_i\{\widehat{\mathrm{GK}}_C(M_i)\}.$$

- *If $MI = 0$ for an ideal $I \lhd A$ then $\widehat{\mathrm{GK}}_C(M_A) = \widehat{\mathrm{GK}}_C(M_{A/I})$.*
- $\widehat{\mathrm{GK}}_C(M) \leq \widehat{\mathrm{GK}}_C(A)$.

- $\widehat{\mathrm{GK}}_C(\sum_{i=1}^r M_i) = \max_i \widehat{\mathrm{GK}}_C(M_i)$.

PROPOSITION 4.4: *Suppose that A and B are affine C-algebras, and that M is an $A - B$-bimodule which is finitely generated as an A-module. Then*

$$\widehat{\mathrm{GK}}_C(M_B) = \widehat{\mathrm{GK}}_C(B/(\mathrm{Ann}\, M_B))$$

and

$$\widehat{\mathrm{GK}}_C(M_B) \le \widehat{\mathrm{GK}}_C({}_AM).$$

Proof: We sketch the proof which is similar to [4, Lemma 5.3]. To prove the first inequality notice that $B/\,\mathrm{Ann}\, M_B$ embeds in the direct sum of a finite number of copies of M, and hence $\widehat{\mathrm{GK}}_C B/\,\mathrm{Ann}\, M_B \le \widehat{\mathrm{GK}}_C M$. The other inequality is clear by Proposition 4.3.

For the second claim, let $B = C\{b_1, \ldots, b_s\}$, $V = 1 \cdot C + b_1 C + \cdots + b_s C$, and $M' = m_1 B + \cdots + m_k B \subseteq M = A\widetilde{m}_1 + \cdots + \widetilde{m}_l$ be a finitely generated submodule of M_B. Further, let $M'' = \sum_i m_i C + \sum \widetilde{m}_i C$. Clearly $M'V^n \subseteq M''V^n$ for all integers $n \ge 0$.

$M''V$ is a finitely generated C-module and since $AM'' \supseteq M''V$ we can find a finitely generated C-module $W \subseteq A$ such that $WM'' \supseteq M''V$. With these choices we have

$$M'V^n \subseteq M''V^n \subseteq W^n M''$$

for any $n \ge 0$, and it follows that $\widehat{\mathrm{GK}}_C M'_B \le \widehat{\mathrm{GK}}_C M''_B \le \widehat{\mathrm{GK}}_C\ {}_AM$ as desired. ∎

COROLLARY 4.5: *Let A and B be affine C-algebras. If ${}_AM_B$ is a bimodule which is finitely generated on both sides, then*

$$\widehat{\mathrm{GK}}_C({}_AM) = \widehat{\mathrm{GK}}_C(M_B),$$

and if furthermore M is faithful on both sides, then

$$\widehat{\mathrm{GK}}_C(A) = \widehat{\mathrm{GK}}_C(B).$$

Now we move on to filtered and graded results. We consider a C-algebra R with an exhaustive filtration $\{F_n R\}_{n \ge 0}$ of C-submodules of R. We have the associated graded algebra $\mathrm{gr}(R) = \bigoplus_{n \ge 0} F_n R / F_{n-1} R$. We single out the so-called natural filtration of an affine algebra $R = C\{a_1, \ldots, a_k\}$, which is obtained by letting $M = 1C + a_1 C + \cdots + a_k C$ and setting $F_n R = M^n$.

Similar definitions apply to modules: Let M be an R-module (R a filtered C-algebra). For a filtration of M $\{F_nM\}_{n\geq 0}$ of C-submodules of M we have the associated graded module $\text{gr}(M) = \bigoplus_{n\geq 0} F_nM/F_{n-1}M$ which is a $\text{gr}(R)$-module. It is standard that M is finitely generated or Noetherian if $\text{gr}(M)$ is. Further we shall have occasion to use the standard filtration of a finitely generated module over an affine algebra. Explicitly, let $M = m_1R + \cdots + m_kR$ and $V = m_1C + \cdots + m_kC$. Now define a filtration of M by $F_nM = VF_nR$ where $\{F_nR\}$ is the natural filtration of R.

For details on filtered modules we refer the reader to [7].

The examples of filtered rings we have in mind are Ore extensions, Weyl algebras, and enveloping algebras. Before giving the explicit results we sketch proofs of generalities on $\widehat{\text{GK}}$ and filtered modules.

Relating $\widehat{\text{GK}}$ of a module to the $\widehat{\text{GK}}$ of its associated graded module is fairly simple. The following is obtained following [4, Chapter 6] mutatis mutandis, noting that reduced rank is additive over short exact sequences.

PROPOSITION 4.6: *Let R be a filtered C-algebra and M an R module. Then*

$$\widehat{\text{GK}}_C \,\text{gr}(M) \leq \widehat{\text{GK}}_C M.$$

The reverse inequality is only true under some additional hypothesis:

PROPOSITION 4.7: *Let R be a filtered affine C-algebra and let M a finitely generated R-module. Then*

$$\widehat{\text{GK}}_C \,\text{gr}(M) = \widehat{\text{GK}}_C M.$$

These results suffice to get some desirable results for enveloping algebras, Weyl algebras, and almost commutative algebras, which we turn to in the next section.

5. Applications

Certainly the value of a generalization of classical GK should be judged on the basis of its applications, to which we now turn our attention. Here we explore how $\widehat{\text{GK}}$ provides natural extensions of Gromov's Theorem concerning group algebras and a theorem of E. Letzter which deals with finite ring extensions and the second layer condition, as well as how the standard results concerning Ore extensions and enveloping algebras may be established without requiring the presence of a field.

Consider the group algebra $C[G]$ for some finitely generated group G. In case C is actually a field, Gromov's Theorem states that the GK dimension of $C[G]$ will be finite if and only if G contains a nilpotent normal subgroup N of finite index. Moreover, in case N is itself a finitely generated group, the precise value of $\widehat{\mathrm{GK}}_C(C[G])$ may be given in terms of the torsionfree ranks of the finitely generated abelian groups which occur in the lower central series of N. These results are easily extended to those base rings C for which $\widehat{\mathrm{GK}}_C(C[G])$ is defined.

THEOREM 5.1 (Gromov):

(1) *Let G be a finitely generated group. Then $\widehat{\mathrm{GK}}_C(C[G]) < \infty$ if and only if G has a nilpotent normal subgroup N of finite index in G.*
(2) *Let N be a finitely generated nilpotent group, with lower central series $N = N_1 \supset N_2 \supset \cdots \supset N_k = 1$. Then*

$$\widehat{\mathrm{GK}}_C(C[N]) = \sum_{j=1}^{k-1} j d_j,$$

where

$$d_j = \mathrm{rank}(N_j/N_{j+1}) = \mathrm{length}_{\mathbb{Q}}(N_j/N_{j+1} \otimes \mathbb{Q}).$$

Proof: (1) We reduce to the case where C is a field. Then $\mathrm{GK}(C[G]) < \infty$, whereupon the standard result may be applied. But this reduction is easy given Proposition 2.2 and the fact that $C[G]$ is a free C-module; if we denote the minimal primes of C by $P_1, P_2, \ldots, P_n$ and the quotient field of C/P_i by L_i, then

$$\begin{aligned}
\widehat{\mathrm{GK}}(C[G]) < \infty &\Leftrightarrow \mathrm{GK}_{L_i}(C[G]/P_i[G] \otimes L_i) < \infty \text{ for all } i \\
&\Leftrightarrow \mathrm{GK}_{L_i}(\frac{C}{P_i}[G] \otimes L_i) < \infty \text{ for all } i \\
&\Leftrightarrow \mathrm{GK}_{L_i}(L_i[G]) < \infty \text{ for all } i \\
&\Leftrightarrow G \text{ has a nilpotent normal subgroup } N\text{of finite index in } G.
\end{aligned}$$

(2) The reduction to the field case is as above, and so the standard result, [4, 11.11], may be applied. ∎

We continue investigating the implications of finite dimensionality by showing that some results of E. Letzter on extensions of Noetherian rings remain valid with GK replaced by $\widehat{\mathrm{GK}}$.

LEMMA 5.2 (after [5, Lemma 3.2]): *Let $A \subseteq B$ be an extension of faithful, affine C-algebras such that B is finitely generated as a right A-module. Assume* *further that A and B are Noetherian, that B is prime, and that B has finite $\widehat{\mathrm{GK}}$. Then A has an Artinian quotient ring, and B is torsionfree as a right A-module.*

Proof: The proof goes through with only one potential point of difficulty. Nevertheless, we sketch the entire argument.

Using the ascending chain condition on $B - A$ sub-bimodules of B, we may obtain a series

$$0 = B_0 \subseteq \cdots \subseteq B_n = B$$

of such sub-bimodules, as well as a set $\{Q_i\}$ of prime ideals of A, in a way which guarantees that each B_i/B_{i-1} is a finitely generated, faithful, and torsionfree $B - A/Q_i$-bimodule (cf. [3, Chapter 7], for instance). Clearly $Q_nQ_{n-1}\cdots Q_1 = (0)$, so that each of the minimal primes of A may be found among the Q_i. Moreover, our "finite $\widehat{\mathrm{GK}}$" assumption allows us to conclude that only minimal primes may be found among the Q_i. Indeed, Corollary 4.5 shows that $\widehat{\mathrm{GK}}_C(B) = \widehat{\mathrm{GK}}_C(A/Q_i)$ for each i; moreover, as each Q_i is the annihilator of some subfactor of B, and each such subfactor is C-faithful, we must have that $Q_i \cap C = (0)$. Now Corollary 2.6 applies, showing that the Q_i are incomparable, and so all minimal.

The result now follows by noting that, if N is the prime radical of A, then since $(B_i/B_{i-1})_A$ is $\mathcal{C}_A(Q_i)$-torsionfree for all i it follows that $(B_i/B_{i-1})_A$ is $\mathcal{C}_A(N)$-torsionfree for all i, and so in fact B_A is itself $\mathcal{C}_A(N)$-torsionfree; therefore $\mathcal{C}_A(N) \subseteq \mathcal{C}_B(0) \cap A \subseteq \mathcal{C}_A(0)$, whence Small's Theorem establishes the existence of an Artinian quotient ring for A (see e.g. [7, Corollary 4.1.4]). ∎

The above Lemma is all that is needed to prove

PROPOSITION 5.3: *Let $A \subseteq B$ be an extension of Noetherian algebras, affine over the center of A, such that B is finitely generated as a right A-module. Suppose further that prime factors B/P of B have finite $\widehat{\mathrm{GK}}$ over the center of their subring $A/(P \cap A)$. Then*

- *If A satisfies the right second layer condition, then so does B.*
- *If A satisfies the right strong second layer condition, then so does B.*
- *If there exists a finite uniform upper bound for the Goldie ranks of prime factors of A, then the same holds true for B.*

Proof: See the proof of [5, Theorem 3.3], and use our Lemma 5.2 in place of Lemma 3.2 employed there. ∎

We next consider Ore extensions where, as one may expect, the $\widehat{\mathrm{GK}}$ of an Ore extension is precisely one more than the $\widehat{\mathrm{GK}}$ of the base ring (at least in the affine case — in the non-affine case the usual problems occur (see [4, 3.9]). The proof here is exactly as given in [4], and relies solely on the additivity of the reduced rank function.

PROPOSITION 5.4 ([4, Proposition 3.5]): *Let A be an affine C-algebra, and suppose that δ is a C-derivation. Then*

$$\widehat{\mathrm{GK}}_C(A[x;\delta]) = \widehat{\mathrm{GK}}_C(A) + 1.$$

Proof: Note first that if M is a finitely generated generating subspace for A, then $M + Cx$ is a finitely generated generating subspace for $A[x;\delta]$. Now for any positive integer n,

$$(M + Cx)^{2n} \supseteq M^n + M^n x + M^n x^2 + \cdots + M^n x^n,$$

whence

$$\rho_C((M + Cx)^{2n}) \geq (n+1)\rho_C(M^n).$$

It follows that

$$\begin{aligned}\widehat{\mathrm{GK}}_C(A[x;\delta]) &= \limsup_n \log_n(\rho_C((M + Cx)^{2n}))\\ &\geq \limsup_n \log_n((n+1)\rho_C(M^n))\\ &= \lim \log_n(n+1) + \limsup_n \log_n(\rho_C(M^n))\\ &= 1 + \widehat{\mathrm{GK}}_C(A).\end{aligned}$$

The reverse inequality is obtained by a similar trick. We claim that for any positive integer n,

$$(M + Cx)^n \subseteq M^{jn} + M^{jn}x + M^{jn}x^2 + \cdots + M^{jn}x^n,$$

where j is the smallest positive integer satisfying $\delta(M) \subseteq M^j$. Clearly this is the case if $n = 0$. Assuming the claim is proven for some fixed value of n, we find that

$$\begin{aligned}(M+Cx)^{n+1} &= M(M+Cx)^n + x(M+Cx)^n\\ &\subseteq \sum_{i=0}^{n} M^{jn+1}x^i + \sum_{i=0}^{n} xM^{jn}x^i\\ &\subseteq \sum_{i=0}^{n+1} M^{j(n+1)}x^i + \sum_{i=0}^{n}(M^{jn}x^{i+1} + \delta(M^{jn})x^i)\\ &\subseteq \sum_{i=0}^{n+1} M^{j(n+1)}x^i + \sum_{i=0}^{n+1} M^{jn}x^i + \sum_{i=0}^{n+1} M^{jn+j-1}x^i\\ &\subseteq \sum_{i=0}^{n+1} M^{j(n+1)}x^i + \sum_{i=0}^{n+1} M^{j(n+1)}x^i + \sum_{i=0}^{n+1} M^{j(n+1)}x^i\\ &= \sum_{i=0}^{n+1} M^{j(n+1)}x^i,\end{aligned}$$

so that the claim is established for any positive exponent. We now get that

$$\begin{aligned}\widehat{\mathrm{GK}}_C(A[x;\delta]) &= \limsup_n \log_n(\rho_C((M+Cx)^n))\\ &\le \limsup_n \log_n(\rho_C((n+1)M^{jn}))\\ &= \lim \log_n(n+1) + \limsup_n \log_n(\rho_C(M^{jn}))\\ &= 1 + \widehat{\mathrm{GK}}_C(A). \qquad \blacksquare\end{aligned}$$

We draw an immediate consequence which could also be established directly:

COROLLARY 5.5: *For the polynomial ring $A = C[x_1, \dots, x_n]$ we have $\widehat{\mathrm{GK}}_C A = n$.*

We now turn to enveloping algebras where we are able to exploit the $\widehat{\mathrm{GK}}$ results on graded modules. Let $\mathfrak{g}$ be a Lie algebra over C and assume that $\mathfrak{g}$ is finitely generated and free as a C-module, say $\mathfrak{g} = \bigoplus_{i=1}^n X_i \cdot C$. Then one can define the universal enveloping $A = U(\mathfrak{g}) = C\langle X_1, \dots, X_n\rangle/(X_iX_j - X_jX_i - [X_i, X_j])$. Let x_i denote the images of X_i in $U(\mathfrak{g})$ and consider the natural filtration of $U(\mathfrak{g})$ by degree in the x_i. The PBW-theorem (see e.g. [2, 2.1.12]) states that the ordered monomials $\{x_1^{i_1} \cdots x_n^{i_n} \mid i_1, \dots, i_n \in \mathbb{N}_0\}$ form a basis for $U(\mathfrak{g})$ as a C-module. It follows the $\mathrm{gr}(U(\mathfrak{g}))$ is a polynomial ring in n variables over C. Thus by 4.7 and 5.5 we get the following:

PROPOSITION 5.6: *With the above notation $\widehat{\mathrm{GK}}_C U(\mathfrak{g}) = n$.*

Further, it is possible to get results for quotients of enveloping algebras as follows: Suppose I is an ideal of $U(\mathfrak{g})$ where $\mathfrak{g}$ is as above, and let M be a finitely generated $U(\mathfrak{g})/I$-module equipped with the standard filtration. Then $\widehat{\mathrm{GK}}_C M = \widehat{\mathrm{GK}}_C \operatorname{gr}(M)$, and since the latter is a finitely generated module over an affine commutative algebra, this number is an integer which cannot exceed n. Thus

PROPOSITION 5.7: *With the above notation, $\widehat{\mathrm{GK}}_C M$ is an integer less than n for any finitely generated $U(\mathfrak{g})/I$-module M.*

Another class of algebras amenable to this type of results is Weyl Algebras. For any base ring C one can consider the nth Weyl algebra

$$A_n(C) = C\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$$

where the relations are that $[x_i, x_j] = [y_i, y_j] = 0$ and $x_i y_j - y_j x_i = \delta_{ij}$. Filtering $A_n(C)$ by degree in the generators it follows that $\operatorname{gr}(A_n(C)) \simeq C[z_1, \ldots, z_{2n}]$ and hence we have

PROPOSITION 5.8: *With the above notation, $\widehat{\mathrm{GK}}_C A_n(C) = 2n$.*

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